Multi-Dimensional Reflective BSDE

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Multi-Dim Reflective BSDE

Lipschitz Growth

m-dim reflective BSDE

$$\begin{cases} Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dB_s + K(T) - K(t); \\ Y(t) \geq L(t), 0 \leq t \leq T, \int_0^T (Y(t) - L(t))' dK(t) = 0. \end{cases} \tag{1}$$

Lipschitz Growth

entry by entry,

$$\begin{cases} Y_1(t) = \xi_1 + \int_t^T g_1(s, Y(s), Z(s))ds - \int_t^T Z_1(s)dB_s + K_1(T) - K_1(t); \\ Y_1(t) \geq L_1(t),\ 0 \leq t \leq T,\ \int_0^T (Y_1(t) - L_1(t))dK_1(t) = 0, \\ \dots \\ Y_m(t) = \xi_m + \int_t^T g_m(s, Y(s), Z(s))ds - \int_t^T Z_m(s)dB_s + K_m(T) - K_m(\\ Y_m(t) \geq L_m(t),\ 0 \leq t \leq T,\ \int_0^T (Y_m(t) - L_m(t))dK_m(t) = 0. \end{cases} \tag{2}$$

Lipschitz Growth

Seek solution (Y, Z, K) in the spaces

$$
\begin{aligned}
& Y = (Y_1, \cdots, Y_m)' \in \mathbb{M}^2(m; 0, T) \\
:= & \{m\text{-dimensional predictable process } \phi \text{ s.t. } \mathbb{E}[\sup_{[0,T]} \phi_t^2] \leq \infty\}; \\
& Z = (Z_1, \cdots, Z_m)' \in \mathbb{L}^2(m \times d; 0, T) \\
:= & \{m \times d\text{-dimensional predictable process } \phi \text{ s.t. } \mathbb{E}[\int_0^T \phi_t^2 dt] \leq \infty\}; \\
& K = (K_1, \cdots, K_m)' = \text{ continuous, increasing process in } \mathbb{M}^2(m; 0, T).
\end{aligned} \tag{3}
$$

Lipschitz Growth

Assumption A 3.1

(1) The random field

$$g = (g_1, \cdots, g_m)' : [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m \quad (4)$$

is predictable in t, and is uniformly Lipschitz in y and z, i.e. there exists a constant $b > 0$, such that

$$|g(t, y, z) - g(t, \bar{y}, \bar{z})| \leq b(\|y - \bar{y}\| + \|z - \bar{z}\|), \forall t \in [0, T]. \quad (5)$$

Further more,

$$\mathbb{E}[\int_0^T g(t, 0, 0)^2 dt] < \infty. \quad (6)$$

(2) The random variable ξ is $\mathscr{F}_T$-measurable and square-integrable. The lower reflective boundary L is progressively measurable, and satisfy $\mathbb{E}[\sup_{[0,T]} L^+(t)^2] < \infty$. $L \leq \xi$, $\mathbb{P}$-a.s.

Lipschitz Growth

Results:

- existence and uniqueness of solution, via Picard iteration
- 1-dim Comparison Theorem (El Karoui et al, 1997)
- continuous dependency property

Linear Growth, Markovian System

l(elle)-dim forward equation

$$\begin{cases} X^{t,x}(s) = x, 0 \leq s \leq t; \\ dX^{t,x}(s) = f(s, X^{t,x}(s))ds + \sigma(s, X^{t,x}(s))dB_s, \ t < s \leq T. \end{cases} \tag{7}$$

m-dim backward equation

$$\begin{cases} Y^{t,x}(s) = \xi(X^{t,x}(T)) + \int_s^T g(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r))dr \\ \qquad - \int_s^T Z^{t,x}(r)dB_r + K^{t,x}(T) - K^{t,x}(s); \\ Y^{t,x}(s) \geq L(s, X^{t,x}(s)), \ t \leq s \leq T, \ \int_t^T (Y^{t,x}(s) - L(s, X^{t,x}(s)))' dK^{t,x}(s) \end{cases} \tag{8}$$

Linear Growth, Markovian System

Assumption A 4.1
(1) Drift $f : [0, T] \times \mathbb{R}^l \to \mathbb{R}^l$, and volatility $\sigma : [0, T] \times \mathbb{R}^l \to \mathbb{R}^{l \times d}$, are deterministic, measurable mappings, locally Lipschitz in x uniformly for all $t \in [0, T]$. And for all $(t, x) \in [0, T] \times \mathbb{R}^l$, $|f(t, x)|^2 + |\sigma(t, x)|^2 \leq C(1 + |x|^2)$, for some constant C.
(2) $g : [0, T] \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m$ is deterministic, measurable, and for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, $|g(t, x, y, z)| \leq b(1 + |x|^p + |y| + |z|)$, for some positive constant b;
(3) for every fixed $(t, x) \in [0, T] \times \mathbb{R}$, $g(t, x, \cdot, \cdot)$ is continuous.
(4) $\xi : \mathbb{R}^l \to \mathbb{R}^m$ deterministic, measurable. $L : [0, T] \times \mathbb{R}^l \to \mathbb{R}^m$ deterministic, measurable, continuous. $\mathbb{E}[\xi(X(T))^2] < \infty$; $\mathbb{E}[\sup_{[0,T]} L^+(t, X(t))^2] < \infty$. $L \leq \xi$, $\mathbb{P}$-a.s.

Linear Growth, Markovian System

Results

- existence of solution, via Lipschitz approximation
- 1-dim Comparison Theorem
- continuous dependency property

What for?

Connections with

- Multi-dim variational inequalities (Feynman-Kac formula)
- Non-zero-sum stoch. differential games (esp. Dynkin games)
- Financial market sensitive to large traders' transactions

References

- Pardoux, E. and S. Peng (1990). Adapted Solution of a Backward Stochastic Differential Equation. *Systems & Control Letters* 14 55-61.
- Hamadène, S., J-P. Lepeltier, and S. Peng (1997). BSDEs with Continuous Coefficients and Stochastic Differential Games. *Pitman Research Notes in Mathematics Series* 364.
- Karatzas, I. and Q. Li (2009). A BSDE Approach to Non-Zero-Sum Stochastic Differential Games of Control and Stopping. *Submitted*. http://stat.columbia.edu/~qinghua

References

- Cvitanić, J., and I. Karatzas (1996). Backward Stochastic Differential Equations with Reflection and Dynkin Games. *The Annals of Probability*. Vol. 24, No. 4, 2024-2056.
- El Karoui, N., C. Kapoudjian, E. Pardoux, S. Peng, M. C. Quenez (1997). Reflected Solutions of Backward SDE'S, and Related Obstacle Problems for PDE's. *The Annals of Probability*. Vol. 25, No. 2, 702-737.
- Hu, Ying, and Shige Peng. On the Comparison Theorem for Multidimensional BSDEs. *C. R. Acad. Sci. Paris, Ser.* I 343 (2006) 135-140.

THAT'S ALL

THANK YOU